

AN OPTIMIZATION PROBLEM IN HEAT CONDUCTION WITH MINIMAL TEMPERATURE CONSTRAINT, INTERIOR HEATING AND EXTERIOR INSULATION

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ABSTRACT. We show the existence and optimal regularity of the optimal temperature configuration in a problem in heat conduction with minimal temperature constraint, interior heating and exterior insulation. Regularity of the two free boundaries is also studied.

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1. INTRODUCTION

In this paper we discuss an optimization problem in heat conduction that may be briefly described as follows: We want to keep the temperature in a room above a given temperature profile using heating sources inside the room and insulation material of a given volume outside the room. The optimal configuration is the one that takes the least energy.

Mathematically, given a bounded smooth domain $D \subset \mathbb{R}^n$, a smooth non-negative function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ compactly supported in D , and a positive number $m > 0$, we seek a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ with $|\{u > 0\} \setminus D| = m$ and $u \geq \phi$. Here $|E|$ denotes the Lebesgue measure of a set E . We also assume $\Delta u = 0$ in $\{u > 0\} \setminus D$ due to insulation, and $\Delta u \leq 0$ in D due to interior heating.

Among this class of functions an optimizer should minimize a certain functional corresponding to the energy taken by the interior heating sources. The most natural functional seems to be the total mass of $-\Delta u$ in D

$$\int_D -\Delta u dx.$$

However, this functional depends on the shape of $\{u > 0\}$ in a highly nonlocal fashion and requires new ideas. Consequently we propose, as a replacement, to study the Dirichlet energy

$$\int \frac{|\nabla u|^2}{2} dx.$$

These two functionals are of the same order.

Intuitively, to save energy, one would like to make u as low as possible subject to $u \geq \phi$, and hence u would solve the obstacle problem in D with ϕ as the obstacle. Now since $0 \leq u \leq \max \phi$ along ∂D and ϕ is compactly supported in D , one has $c \leq \phi \leq C$ in the contact set $\{u = \phi\}$ for

some c and C depending only on ϕ and D . Hence with Gauss-Green theorem and the fact that $-\Delta u$ is supported in the contact set, one has the following formal calculation

$$\begin{aligned} \int \frac{|\nabla u|^2}{2} dx &= \int_{\{u>0\}} \frac{|\nabla u|^2}{2} dx \\ &= \int_{\{u>0\}} -u\Delta u/2 dx \\ &= \int_{\{u=\phi\}} -u\Delta u dx \\ &\sim \int_{\{u=\phi\}} -\Delta u dx \\ &= \int_D -\Delta u dx. \end{aligned}$$

As a result, we propose to study the following optimization problem:

Physical Problem: Find a minimizer of the Dirichlet energy

$$\int \frac{|\nabla u|^2}{2} dx$$

over $K_0 = \{u \in H_0^1(\mathbb{R}^n) : u \geq \phi, |\{u > 0\} \setminus D| = m, \Delta u \leq 0 \text{ in } D, \Delta u = 0 \text{ in } \{u > 0\} \setminus D\}$.

Here the inequalities on Δu are understood in the distributional sense.

Concerning the minimizer our main result is the existence and optimal regularity

Theorem 1.1. *There exists a minimizer to the Physical Problem. This minimizer is Lipschitz continuous in \mathbb{R}^n .*

There are two free boundaries coming from the interior contact set and the exterior boundary $\partial\{u > 0\}$. These correspond to the boundary of effective heating sources and the boundary of insulation material, respectively. Concerning the regularity of the interior free boundary we establish

Theorem 1.2. *For $\Delta\phi$ uniformly negative in $\{\phi > 0\}$, the interior free boundary $\partial(\{u > \phi\} \cap D)$ is smooth except on a set of singular points, which are covered by a countable union of lower-dimensional C^1 manifolds.*

Concerning the exterior free boundary we have

Theorem 1.3. *The exterior free boundary $\partial\{u > 0\}$ is smooth except on a H^{n-1} -null set.*

Here H^{n-1} is the $(n-1)$ -dimensional Hausdorff measure.

Similar problems have been studied by Alt-Caffarelli [2], Aguilera-Alt-Caffarelli [1], Aguilera-Caffarelli-Spruck [3] and Teixeira [7], where the authors studied various functionals that are of the same order of the Dirichlet energy. The results and techniques in this paper are very much inspired by these previous work. However there are also significant differences.

On the physical level, instead of prescribing temperature along the walls of the room as in previous works, we consider a minimal temperature profile in the interior of the room. This changes the problem from a boundary value problem in D^c to a problem in the entire \mathbb{R}^n .

This leads to some new difficulties, the most fundamental one being the sign-changing Δu . In all previous works, u is a subsolution throughout the domain of concern, which is the source of regularity of the minimizer. Here, however, Δu changes signs. We get around this by studying a series of perturbed problems. These perturbed problems obtain very regular solutions that converge to a minimizer of our problem. Also Lipschitz regularity is persistent along this limiting process, which gives the optimal regularity of the minimizer.

Once the optimal regularity of the minimizer is established, the problem naturally splits into an interior obstacle problem and an exterior one-phase problem. This allows us to use previous results and hence establish the regularity of the two free boundaries.

This paper is organized as follows: in Section 2 we introduce a three-parameter-family of perturbed problems. We show the existence and first properties of minimizers to these problems. In Section 3 we give estimates uniform in one of the parameters. This is exploited in Section 4 to obtain ‘asymptotic’ minimizers of a two-parameter family of perturbed problems. Estimates uniform in one of remaining two parameters is also established in Section 4. This gives rise to limiting solutions to yet another family of perturbed problems, which now depend on one parameter. In Section 5. we study the free boundary regularity of these limiting solutions. In the last section, Section 6, we connect this one-parameter perturbed problem to our original Physical Problem by showing that when the last parameter is small enough, a minimizer of this perturbed problem actually solves our Physical Problem. This completes the proof for main results as estimates on minimizers of the perturbed problems apply to a minimizer to the Physical Problem. We also show that the positive phases of these minimizers are well-localized in a bounded set, hence any local estimates is actually uniform over the domain, and that the optimization in \mathbb{R}^n is the same as in a big but bounded set.

2. A THREE-PARAMETER FAMILY OF PERTURBED PROBLEMS

For small positive parameters κ_1 , κ_2 and ϵ , we define the following functions:

- $A_{\kappa_1} : \mathbb{R} \rightarrow \mathbb{R}$ is a nonnegative decreasing convex function that vanishes on $[0, +\infty)$. It equals $-\frac{1}{\kappa_1}(t - \frac{\kappa_1}{2})$ for $t < -\kappa_1$, and smoothly interpolates between $-\kappa_1$ and 0.
 α_{κ_1} is the derivative of A_{κ_1} .
- $B_{\kappa_2} : \mathbb{R} \rightarrow \mathbb{R}$ is a piecewise linear function that vanishes on $(-\infty, 0]$ and equals 1 on $[\kappa_2, +\infty)$.
 β_{κ_2} is the derivative of B_{κ_2} .
- $f_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ is the piecewise linear function that equals 0 at m , has slope $\frac{1}{\epsilon}$ to the right of m , and slope ϵ to the left of m .

We study the following three-parameter functional

$$(2.1) \quad J_{\kappa_1, \kappa_2, \epsilon}(w) = \int \frac{|\nabla w|^2}{2} dx + A_{\kappa_1}(w - \phi) + f_\epsilon\left(\int_{D^c} B_{\kappa_2}(u)\right)$$

over all $H_0^1(\mathbb{R}^n)$ functions.

Remark 2.1. We enlarge the class of functions under consideration from K_0 to all $H_0^1(B_R)$ functions. To obtain solutions to our original problem we impose three-parameter penalization/ regularization. A_{κ_1} is to penalize functions that do not lie above ϕ , which seems a standard technique in the study of obstacle-type problems [6]. B_{κ_2} is to regularize $u \mapsto |\{u > 0\} \setminus D|$ as in Caffarelli-Salsa [4]. f_ϵ is to penalize functions with the wrong volume of positive phase [1].

Remark 2.2. We would often suppress subscripts when there is no ambiguity.

The following gives a competitor that may be far from optimal, but it is universal in the sense that it gives estimates independent of all parameters.

Proposition 2.3. *There is $M = M(\phi, D)$ and $w \in K_0$ such that*

$$J_{\kappa_1, \kappa_2, \epsilon}(w) = M$$

for all positive κ_1, κ_2 and ϵ .

Proof. Let w be the minimizer of the Dirichlet energy among all $H_0^1(D)$ functions above ϕ . Then $A(w - \phi)$ is constantly 0. $B_{\kappa_2}(w)$ vanishes outside D thus

$$J_{\kappa_1, \kappa_2, \epsilon}(w) = \int \frac{|\nabla w|^2}{2} dx =: M$$

independent of the parameters.

Obviously $w \geq \phi$. $\Delta w \leq 0$ in D as a standard result from obstacle problem. $\{w > 0\} \setminus D = \emptyset$. Thus $w \in K_0$. \square

Next we establish the existence of minimizers to the perturbed problems.

Proposition 2.4. *There exist minimizers to the perturbed functionals.*

Proof. The function in Proposition 2.3 is a competitor and shows the functional is not always infinite.

Then the existence follows from a standard argument using the direct method [2]. \square

The next proposition gives an \mathcal{L}^∞ estimate on minimizers, and shows we only need to consider functions with one phase.

Proposition 2.5. *If $u_{\kappa_1, \kappa_2, \epsilon}$ is a minimizer to $J_{\kappa_1, \kappa_2, \epsilon}$, then*

$$0 \leq u_{\kappa_1, \kappa_2, \epsilon} \leq \max \phi.$$

Proof. Use $u_{\kappa_1, \kappa_2, \epsilon} - t \min(u, 0)$ and $u_{\kappa_1, \kappa_2, \epsilon} - t \min(u - \max \phi, 0)$ as competitors and study the first order behavior as $t \rightarrow 0^+$. \square

Proposition 2.6. *If $u_{\kappa_1, \kappa_2, \epsilon}$ is a minimizer to $J_{\kappa_1, \kappa_2, \epsilon}$, then*

$$(2.2) \quad \Delta u_{\kappa_1, \kappa_2, \epsilon} = \alpha_{\kappa_1}(u_{\kappa_1, \kappa_2, \epsilon} - \phi) + f' \left(\int_{D^c} B_{\kappa_2}(u_{\kappa_1, \kappa_2, \epsilon}) \right) \beta_{\kappa_2}(u_{\kappa_1, \kappa_2, \epsilon}) \chi_{D^c}.$$

Proof. This is the Euler-Lagrange equation of the perturbed energy functional. \square

3. SENDING $\kappa_1 \rightarrow 0$

In this section we give uniform $C^{1, \alpha}$ -estimate of minimizers independent of κ_1 , establishing compactness when $\kappa_1 \rightarrow 0$. The limiting function lies above ϕ and asymptotically minimizes a two-parameter functional.

With standard regularity theory for elliptic equations, Proposition 2.6 gives $C^{1, \alpha}$ and $W^{2, p}$ regularity of the minimizers for any $0 < \alpha < 1$ and $1 \leq p < +\infty$. The goal, however, is to establish estimates independent of κ_1 . This begins with an uniform estimates on $\|\alpha_{\kappa_1}(u - \phi)\|_\infty$.

Proposition 3.1. $\|\alpha_{\kappa_1}(u - \phi)\|_\infty \leq \|\phi\|_{C^{1, 1}} + \frac{1}{\epsilon \kappa_2}.$

Proof. Define $\tilde{u} = u - \phi$, then the equation for \tilde{u} is

$$\Delta \tilde{u} = \alpha_{\kappa_1}(\tilde{u}) + f' \left(\int_{D^c} B(u) \right) \beta(u) \chi_{D^c} - \Delta \phi.$$

Since for each fixed $\kappa_1 > 0$ α_{κ_1} is a bounded smooth function, $\alpha(\tilde{u})^p$ can be used as a test function for this equation:

$$0 = \int \nabla \tilde{u} \cdot (p \alpha(\tilde{u})^{p-1} \alpha'(\tilde{u}) \nabla \tilde{u}) + \alpha(\tilde{u})^{p+1} + f' \left(\int_{D^c} B(u) \right) \beta(u) \chi_{D^c} \alpha(\tilde{u})^p - \Delta \phi \alpha(\tilde{u})^p.$$

If we choose p to be even, then the first two terms are negative due to monotonicity and convexity of A . Consequently one has

$$\begin{aligned} \int_D |\alpha(\tilde{u})|^{p+1} &\leq \int_D f' \left(\int_{D^c} B(u) \right) \beta(u) \chi_{D^c} \alpha(\tilde{u})^p - \Delta \phi \alpha(\tilde{u})^p \\ &\leq \left(\int_D |f' \left(\int_{D^c} B(u) \right) \beta(u) + \Delta \phi|^p \right)^{1/p} \left(\int_D |\alpha(\tilde{u})|^{p+1} \right)^{\frac{p}{p+1}}. \end{aligned}$$

Note that we used the fact that $\alpha(\tilde{u})$ is supported in D since $u \geq \phi$ outside D .

As a result, $\|\alpha(\tilde{u})\|_{\mathcal{L}^{p+1}(D)} \leq (\|\phi\|_{C^{1,1}} + \frac{1}{\epsilon \kappa_2}) |D|^{1/p}$. Normalizing the Lebesgue measure gives

$$\left(\int_D |\alpha(\tilde{u})|^{p+1} \frac{dx}{|D|} \right)^{\frac{1}{p+1}} \leq (\|\phi\|_{C^{1,1}} + \frac{1}{\epsilon \kappa_2}) |D|^{\frac{1}{p} - \frac{1}{p+1}}.$$

$p \rightarrow +\infty$ gives the desired estimate. □

The proposition above says the right-hand side of (2.2) is bounded independent of κ_1 , which gives uniform $C^{1,\alpha}$ -estimate of minimizers.

Theorem 3.2. *Let $u_{\kappa_1, \kappa_2, \epsilon}$ be a minimizer of $J_{\kappa_1, \kappa_2, \epsilon}$, then for any compact set K and $0 < \alpha < 1$ one has*

$$\|u_{\kappa_1, \kappa_2, \epsilon}\|_{C^{1,\alpha}(K)} \leq C(K, \alpha, n) (\|\phi\|_{C^{1,1}} + \frac{1}{\epsilon \kappa_2}).$$

The following is a direct consequence via Arzela-Ascoli:

Corollary 3.3. *Up to a subsequence $\kappa_1 \rightarrow 0$, u_{κ_1} converges to some u weakly in $H_0^1(\mathbb{R}^n)$ and locally uniformly in $C^{1,\alpha}(\mathbb{R}^n)$.*

With the uniform bound on energy as in Proposition 2.3, the limit u lies above our obstacle:

Proposition 3.4. $u \geq \phi$.

Proof. For given $\delta > 0$ and compact K , $\{u - \phi < -\delta\} \cap K$ is contained in $\{u_{\kappa_1} - \phi < -\delta/2\} \cap K$ for small κ_1 . For the latter set one has the following estimate

$$M \geq \int A_{\kappa_1}(u_{\kappa_1} - \phi) dx \geq \frac{1}{\kappa_1} \delta/2 |\{u_{\kappa_1} - \phi < -\delta/2\} \cap K|.$$

Taking $\kappa_1 \rightarrow 0$ forces $\{u_{\kappa_1} - \phi < -\delta/2\} \cap K$ to be null. □

4. SENDING $\kappa_2 \rightarrow 0$

Now we define a new two-parameter family of perturbed functionals that do not involve κ_1 anymore:

$$(4.1) \quad J_{\kappa_2, \epsilon}(w) = \int \frac{|\nabla w|^2}{2} dx + f_\epsilon \left(\int_{D^c} B_{\kappa_2}(w) \right).$$

Ideally we would expect the limit u from Corollary 3.3 to be a minimizer to this new functional over functions that lie above ϕ . However this is not always true due to the lack of convexity in $u \mapsto f_\epsilon(\int_{D^c} B_{\kappa_2}(u))$. Nevertheless we show that u minimizes the energy ‘asymptotically’ as in the next lemma. It is a variation of the classical lemma of Minty [5] applied to an operator with a monotone part $u \mapsto \Delta u$ and a regular part $u \mapsto \beta(u)$.

Lemma 4.1. *Let u be as in Corollary 3.3, then for any $v \in H_0^1(\mathbb{R}^n)$ with $v \geq \phi$, one has*

$$(4.2) \quad \left. \frac{d}{d\lambda} \right|_{\lambda=0^+} J_{\kappa_2, \epsilon}(u + \lambda(v - u)) \geq 0.$$

Proof. Since u_{κ_1} is a minimizer of $J_{\kappa_1, \kappa_2, \epsilon}$, for any $v \in H_0^1$ and $t > 0$ one has

$$\begin{aligned} \int \frac{|\nabla u_{\kappa_1} + t\nabla(v - u_{\kappa_1})|^2}{2} + A(u_{\kappa_1} + t(v - u_{\kappa_1}) - \phi) + f\left(\int_{D^c} B(u_{\kappa_1} + t(v - u_{\kappa_1}))\right) \\ \geq \int \frac{|\nabla u_{\kappa_1}|^2}{2} + A(u_{\kappa_1} - \phi) + f\left(\int_{D^c} B(u_{\kappa_1})\right). \end{aligned}$$

Thus one has a sign on the first order term

$$\int \nabla u_{\kappa_1} \cdot \nabla(v - u_{\kappa_1}) + \alpha(u_{\kappa_1} - \phi)(v - u_{\kappa_1}) + f'\left(\int_{D^c} B(u_{\kappa_1})\right) \int_{D^c} \beta(u_{\kappa_1})(v - u_{\kappa_1}) \geq 0.$$

Note that we have the following monotonicity for the first two terms of the operator, coming from the monotonicity of the Dirichlet energy and the function α :

$$\int (\nabla v - \nabla u_{\kappa_1}) \cdot \nabla(v - u_{\kappa_1}) + (\alpha(v - \phi) - \alpha(u_{\kappa_1} - \phi))(v - u_{\kappa_1}) \geq 0.$$

Combining the two inequalities above we have for any $v \in H_0^1$

$$\int \nabla v \cdot \nabla(v - u_{\kappa_1}) + \alpha(v - \phi)(v - u_{\kappa_1}) + f'\left(\int_{D^c} B(u_{\kappa_1})\right) \int_{D^c} \beta(u_{\kappa_1})(v - u_{\kappa_1}) \geq 0.$$

And in particular for $v \geq \phi$

$$\int \nabla v \cdot \nabla(v - u_{\kappa_1}) + f'\left(\int_{D^c} B(u_{\kappa_1})\right) \int_{D^c} \beta(u_{\kappa_1})(v - u_{\kappa_1}) \geq 0.$$

Due to weak convergence in H_0^1 of $u_{\kappa_1} \rightarrow u$,

$$\int \nabla v \cdot \nabla(v - u_{\kappa_1}) \rightarrow \int \nabla v \cdot \nabla(v - u).$$

The rest of the terms are more regular and we have the following

$$\begin{aligned} \left| \int_{D^c} \beta(u_{\kappa_1})(v - u_{\kappa_1}) - \int_{D^c} \beta(u)(v - u) \right| &\leq \left| \int_{D^c} \beta(u_{\kappa_1})(u_{\kappa_1} - u) \right| + \left| \int_{D^c} (\beta(u_{\kappa_1}) - \beta(u))(v - u) \right| \\ &\leq C(\kappa_2) \|u_{\kappa_1} - u\|_{\mathcal{L}^2} + \left| \int_{D^c} (\beta(u_{\kappa_1}) - \beta(u))(v - u) \right| \\ &= o(1) + \left| \int_{D^c} (\beta(u_{\kappa_1}) - \beta(u))(v - u) \right|. \end{aligned}$$

It remains to show $\left| \int_{D^c} (\beta(u_{\kappa_1}) - \beta(u))(v - u) \right| \rightarrow 0$. To this end, note that $(\beta(u_{\kappa_1}) - \beta(u))$ is bounded and $u - v \in \mathcal{L}^2$, thus for any given $\delta > 0$ we can find R big enough so that

$$\left| \int_{D^c} (\beta(u_{\kappa_1}) - \beta(u))(v - u) - \int_{D^c \cap B_R} (\beta(u_{\kappa_1}) - \beta(u))(v - u) \right| \leq \delta.$$

Now on the compact set B_R one can apply bounded convergence theorem to show

$$\int_{D^c \cap B_R} (\beta(u_{\kappa_1}) - \beta(u))(v - u) \rightarrow 0.$$

As a result one has the desired estimate

$$\int \nabla v \cdot \nabla(v - u) + f'\left(\int_{D^c} B(u)\right) \int_{D^c} \beta(u)(v - u) \geq 0.$$

□

Let $u_{\kappa_1, \kappa_2, \epsilon}$ be a minimizer of $J_{\kappa_1, \kappa_2, \epsilon}$ and u_{κ_2} be the limit when $\kappa_1 \rightarrow 0$ as in Corollary 3.3, we now begin the program of sending $\kappa_2 \rightarrow 0$. For this one needs estimate on u_{κ_2} uniform in κ_2 . The previous lemma gives the equation for u_{κ_2} :

Corollary 4.2.

$$(4.3) \quad \Delta u = f'(\int_{D^c} B(u))\beta(u)\chi_{D^c} \text{ in } \{u > \phi\}.$$

$$(4.4) \quad -(\|\phi\|_{C^{1,1}} + \frac{1}{\epsilon\kappa_2}) \leq \Delta u \leq f'(\int_{D^c} B(u))\beta(u)\chi_{D^c} \text{ in } \mathbb{R}^n.$$

Proof. Equation (4.3) and the right-hand side of (4.4) are direct consequence of the previous lemma.

The left-hand side of (4.4) comes from the weak convergence of $u_{\kappa_1} \rightarrow u$ in H_0^1 and the uniform bound on the right-hand side of (2.2). \square

The domain naturally splits into three regions: $\{u \leq \kappa_2\}$, $\{u > \kappa_2\} \cap \{u > \phi\}$ and $\{u = \phi\}$. In the first region we have smallness of data, in the second u is harmonic, and in the last u induces regularity from the obstacle. The following propositions establish estimates in these regions.

Proposition 4.3. *If $x_0 \in \{u \leq \kappa_2\}$, then*

$$|\nabla u(x_0)| \leq C(n)(\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon} + 1).$$

Proof. Define

$$w(y) = \frac{1}{\kappa_2}u(x_0 + \kappa_2 y).$$

Then

$$-(\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon}) \leq \Delta w \leq \frac{1}{\epsilon}.$$

Also $w(0) \leq 1$.

Being a nonnegative function with bounded Laplacian, $|w| \leq C(n)(\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon} + 1)$ in B_1 . Thus $|\nabla w(0)| \leq C(n)(\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon} + 1)$ by standard interior estimates for elliptic equations.

Note $\nabla w(0) = \nabla u(x_0)$ one sees the desired estimate. \square

Proposition 4.4. *If $x_0 \in \{u = \phi\}$, then*

$$|\nabla u(x_0)| \leq C(n)(\|\phi\|_{C^1} + \|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon} + 1).$$

Proof. Define

$$w(y) = \frac{1}{\kappa_2}(u(x_0 + \kappa_2 y) - \phi(x_0 + \kappa_2 y)).$$

Then $w \geq 0$ and $w(0) = 0$. Moreover,

$$-(2\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon}) \leq \Delta w \leq (\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon}).$$

Note $\nabla u(x_0) = \nabla w(0) + \nabla \phi(x_0)$, the estimate follows from elliptic regularity as in the previous proposition. \square

Proposition 4.5. *If $x_0 \in \{u > \phi\} \cap \{u > \kappa_2\}$, then*

$$|\nabla u(x_0)| \leq C(n)(\|\phi\|_{C^1} + \|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon}).$$

Proof. Let $d = \text{dist}(x_0, \partial(\{u > \phi\} \cap \{u > \kappa_2\}))$, and $y_0 \in \partial(\{u > \phi\} \cap \{u > \kappa_2\})$ be such that $|y_0 - x_0| = d$. Then in particular $u(y_0) = \kappa_2$ or $u(y_0) = \phi(y_0)$.

If $u(y_0) = \kappa_2$, we define

$$w(y) = \frac{1}{d}(u(x_0 + dy) - \kappa_2).$$

Then one has

$$\Delta w = 0 \text{ in } B_1,$$

$$w \geq 0 \text{ in } B_1,$$

$$w(\tilde{y}_0) = 0$$

and

$$|\nabla w(\tilde{y}_0)| \leq C(n)(\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon} + 1).$$

Here \tilde{y}_0 is the point on ∂B_1 corresponding to y_0 . The last estimate comes from Proposition 4.3.

By Harnack, $w(y) \geq c(n)w(0)$ in $B_{1/2}$. Define a scaled fundamental solution

$$\Psi(y) = \frac{cw(0)}{2^{n-2}-1}(\frac{1}{|y|^{n-2}} - 1),$$

then $\Psi = 0$ on ∂B_1 , $\Psi = cw(0)$ along $\partial B_{1/2}$ and $\Delta \Psi = 0$ in $B_1 \setminus B_{1/2}$. Comparison principle then gives $\Psi \leq w$ in $B_1 \setminus B_{1/2}$.

However, $w(\tilde{y}_0) = \Psi(\tilde{y}_0)$ thus $\nabla w(\tilde{y}_0) \cdot n \geq \nabla \Psi(\tilde{y}_0) \cdot n$, where n is the inner normal vector to B_1 at \tilde{y}_0 .

This gives

$$C(n)(\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon} + 1) \geq |\nabla w(\tilde{y}_0)| \geq \nabla \Psi(\tilde{y}_0) \cdot n = c(n)w(0).$$

Then $w(y) \leq C(n)w(0) \leq C(n)(\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon} + 1)$ in $B_{1/2}$ by Harnack, and elliptic regularity gives

$$|\nabla u(x_0)| = |\nabla w(0)| \leq C(n)(\|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon} + 1).$$

For the case when $y_0 \in \{u = \phi\}$, we define $w(y) = \frac{1}{d}(u(x_0 + dy) - \phi(x_0 + dy))$, which is another nonnegative harmonic function in B_1 that vanishes at one point on ∂B_1 where we have a gradient estimate from Proposition 4.4. Thus similar barrier argument applies. \square

We collect these results to get the uniform Lipschitz estimate independent of κ_2 :

Theorem 4.6. *Let u_{κ_2} be the limit as in Corollary 3.3 of u_{κ_1, κ_2} as $\kappa_1 \rightarrow 0$. Then*

$$|\nabla u_{\kappa_2}| \leq C(n)(\|\phi\|_{C^1} + \|\phi\|_{C^{1,1}}\kappa_2 + \frac{1}{\epsilon}).$$

Remark 4.7. Since there is a jump in gradient along $\partial\{u > 0\}$, this Lipschitz estimate is the optimal regularity. See [2].

Again by Arzela-Ascoli we have the following

Corollary 4.8. *Let $u_{\kappa_1, \kappa_2, \epsilon}$ be a minimizer of $J_{\kappa_1, \kappa_2, \epsilon}$. Then up to a subsequence as $\kappa_1, \kappa_2 \rightarrow 0$, $u_{\kappa_1, \kappa_2, \epsilon} \rightarrow u_\epsilon$ weakly in H_0^1 and locally uniformly in C^α for any $0 < \alpha < 1$. Moreover*

$$|\nabla u_\epsilon| \leq C(n)(\|\phi\|_{C^1} + 1/\epsilon).$$

5. REGULARITY OF FREE BOUNDARIES

Now we can define a family of perturbed functionals with only one parameter:

$$(5.1) \quad J_\epsilon(w) = \int \frac{|\nabla w|^2}{2} + f_\epsilon(|\{w > 0\} \setminus D|).$$

Our first result is the minimality of u :

Theorem 5.1. *Let u_ϵ be as in Corollary 4.8. Then it is a local minimizer of J_ϵ over functions above ϕ .*

Proof. Suppose, on the contrary, that there is $\delta > 0$, and $v \geq \phi$ with $v - u$ supported in $B_r(x_0)$ such that

$$(5.2) \quad \int_{B_r(x_0)} \frac{|\nabla v|^2}{2} + f\left(\int_{D^c \cap B_r(x_0)} \chi_{\{v>0\}}\right) < \int_{B_r(x_0)} \frac{|\nabla u|^2}{2} + f\left(\int_{D^c \cap B_r(x_0)} \chi_{\{u>0\}}\right) - \delta.$$

Since $B_{\kappa_2}(v) \rightarrow \chi_{\{v>0\}}$ as $\kappa_2 \rightarrow 0$, the left-hand side of (5.2) satisfies the following

$$\begin{aligned} \int_{B_r(x_0)} \frac{|\nabla v|^2}{2} + f\left(\int_{D^c \cap B_r(x_0)} \chi_{\{v>0\}}\right) &= \lim_{\kappa_2 \rightarrow 0} \int_{B_r(x_0)} \frac{|\nabla v|^2}{2} + f\left(\int_{D^c \cap B_r(x_0)} B_{\kappa_2}(v)\right) \\ &= \lim_{\kappa_1, \kappa_2 \rightarrow 0} \int_{B_r(x_0)} \frac{|\nabla v|^2}{2} + A_{\kappa_1}(v - \phi) + f\left(\int_{D^c \cap B_r(x_0)} B_{\kappa_2}(v)\right). \end{aligned}$$

Meanwhile, if we fix a small $\gamma > 0$, then the right-hand side of (5.2) satisfies

$$\begin{aligned} \int_{B_r(x_0)} \frac{|\nabla u|^2}{2} + f\left(\int_{D^c \cap B_r(x_0)} \chi_{\{u>0\}}\right) - \delta &\leq \int_{B_r(x_0)} \frac{|\nabla u|^2}{2} + f\left(\int_{D^c \cap B_r(x_0) \cap \{u \geq \gamma\}} \chi_{\{u>0\}}\right) - \delta/2 \\ &\leq \lim_{\kappa_2 \rightarrow 0} \int_{B_r(x_0)} \frac{|\nabla u_{\kappa_2}|^2}{2} + f\left(\int_{D^c \cap B_r(x_0) \cap \{u \geq \gamma\}} \chi_{\{u_{\kappa_2}>0\}}\right) - \delta/2 \\ &\leq \lim_{\kappa_2 \rightarrow 0} \int_{B_r(x_0)} \frac{|\nabla u_{\kappa_2}|^2}{2} + f\left(\int_{D^c \cap B_r(x_0) \cap \{u \geq \gamma\}} B_{\kappa_2}(u_{\kappa_2})\right) - \delta/2 \\ &\leq \lim_{\kappa_1, \kappa_2 \rightarrow 0} \int_{B_r(x_0)} \frac{|\nabla u_{\kappa_1, \kappa_2}|^2}{2} + f\left(\int_{D^c \cap B_r(x_0) \cap \{u \geq \gamma\}} B_{\kappa_2}(u_{\kappa_1, \kappa_2})\right) - \delta/2 \\ &\leq \lim_{\kappa_1, \kappa_2 \rightarrow 0} \int_{B_r(x_0)} \frac{|\nabla u_{\kappa_1, \kappa_2}|^2}{2} + A_{\kappa_1}(u_{\kappa_1, \kappa_2} - \phi) \\ &\quad + f\left(\int_{D^c \cap B_r(x_0)} B_{\kappa_2}(u_{\kappa_1, \kappa_2})\right) - \delta/2. \end{aligned}$$

Here we used Fatou's lemma and the fact that $B_{\kappa_2}(u_{\kappa_2}) = \chi_{\{u_{\kappa_2}>0\}}$ on $\{u > \gamma\}$ as long as κ_2 is small enough.

Combining these inequalities we could conclude that

$$J_{\kappa_1, \kappa_2}(v) < J_{\kappa_1, \kappa_2}(u_{\kappa_1, \kappa_2}) - \delta/4$$

for small κ_1, κ_2 , contradicting the minimality of u_{κ_1, κ_2} . \square

As a simple corollary we have the Euler-Lagrange equation satisfied by u :

Corollary 5.2.

$$\begin{aligned} \Delta u &\leq 0 \text{ in } D. \\ \Delta u &= 0 \text{ in } D \cap \{u > \phi\}. \\ \Delta u &\geq 0 \text{ in } D^c. \\ \Delta u &= 0 \text{ in } \{u > 0\} \setminus D. \end{aligned}$$

Also u minimizes the Dirichlet energy over $K_1 := \{w \in H^1(D) | w \geq \phi, w = u \text{ on } \partial D\}$, thus u is an obstacle solution in D with ϕ as obstacle and $u|_{\partial D}$ as boundary data. Therefore the standard theory of obstacle problem applies and gives the regularity of the interior free boundary $\partial(\{u > \phi\} \cap D)$ [6]:

Theorem 5.3. *For $\Delta \phi$ uniformly negative in $\{\phi > 0\}$, the interior free boundary $\partial(\{u > \phi\} \cap D)$ is smooth except on a set of singular points, which are covered by a countable union of lower dimensional C^1 -manifolds.*

Regularity of the exterior free boundary $\partial\{u > 0\}$ begins with the following non-degeneracy lemma, which can be proved with the same techniques as in Lemma 3.4 of [2]:

Lemma 5.4. *There is $c = c(n)$ such that $u = 0$ in $B_{R/2}(x_0)$ whenever*

$$\frac{1}{R \cdot H^{n-1}(\partial B_R)} \int_{\partial B_R(x_0)} u dH^{n-1} < c(n)\epsilon$$

and $B_R(x_0) \subset (D \cap \{u = \phi\})^c$.

This along with the uniform Lipschitz estimate gives the following lower density estimate of the positive phase:

Lemma 5.5. *For $x_0 \in \overline{\{u > 0\}}$ and $B_R(x_0) \subset (D \cap \{u = \phi\})^c$, then*

$$\frac{|B_R \cap \{u > 0\}|}{|B_R|} \geq \frac{c(n)}{\|\phi\|_{C^1} + 1/\epsilon} \epsilon.$$

Proof. By the previous lemma there is $y_0 \in \partial B_{R/2}(x_0)$ such that $u(y_0) > c(n)\epsilon R$. By Lipschitz continuity $u(y) > 0$ if $|y - y_0| \leq c(n)\epsilon R / \text{Lip}(u)$. \square

Note that as long as we do not touch the interior contact set, we have all ingredients for the theory of harmonic functions with linear growth as in [2]. Consequently we can use Theorem 4.5 and 4.8 there to obtain the following structure theorem:

Theorem 5.6. *$H^{n-1}(K \cap \partial\{u > 0\}) < \infty$ for any compact K .*

There is a Borel q_u such that $\Delta u|_{(D \cap \{u = \phi\})^c} = q_u H^{n-1}|_{\partial\{u > 0\}}$.

For any compact set K there are $0 < c(n, \|\phi\|_{C^1}, K, \epsilon) \leq C(n, \|\phi\|_{C^1}, K, \epsilon) < \infty$ such that for $x_0 \in \partial\{u > 0\}$ and $B_r(x_0) \subset (D \cap \{u = \phi\})^c$ one has

$$c \leq q_u(x_0) \leq C$$

and

$$cr^{n-1} \leq H^{n-1}(B_r(x_0) \cap \partial\{u > 0\}) \leq Cr^{n-1}.$$

For H^{n-1} -almost every x_0 in $\partial\{u > 0\}$,

$$u(x_0 + x) = q_u(x_0) \max\{-x \cdot \nu(x_0), 0\} + o(|x|)$$

where $\nu(x_0)$ is the outer normal to the reduced boundary of $\{u > 0\}$ at x_0 .

Moreover with techniques from [1] one see the following:

Theorem 5.7. *q_u is constant H^{n-1} almost everywhere on $\partial\{u > 0\}$.*

From here the theory of weak solutions in [2] can be applied to obtain the following:

Theorem 5.8. *$\partial\{u > 0\}$ is smooth except on a H^{n-1} -null set.*

6. CONNECTION TO THE PHYSICAL PROBLEM

In this section we show that for small ϵ a solution to the one-parameter perturbed problem actually solves the original Physical Problem. To this end we first have the following:

Proposition 6.1. *For small $\epsilon > 0$, $|\{u > 0\} \setminus D| = m$.*

Proof. Note that in the exterior domain D^c our solution solves the problem in [1] with a Lipschitz boundary datum. Hence Theorem 7 there can be applied. \square

Remark 6.2. In particular we do not need to send $\epsilon \rightarrow 0$.

We collect results on u_ϵ for small ϵ as in the previous proposition to obtain the following:

Theorem 6.3. *For $\epsilon > 0$ small, u_ϵ optimizes the Physical Problem.*

Proof. Since $|\{u > 0\} \setminus D| = m$, $f(|\{u > 0\} \setminus D|)$ vanishes. Thus u is a minimizer for the Dirichlet energy over functions above ϕ .

Also Corollary 5.2 establishes right signs on the Laplacian of u . Hence $u \in K_0$. \square

The next theorem states that positive phase is well localized inside a bounded set. As a result, outside the interior contact set, any local estimate can be upgraded to global estimate with constants independent of the compact set. Also, the optimization in \mathbb{R}^n is actually the same as in a big but bounded set.

Theorem 6.4. *Let u be a minimizer, then*

$$\text{diam}(\{u > 0\}) \leq \text{diam}(D) + 1 + C(n)m \frac{\|\phi\|_{C^1} + 1/\epsilon}{\epsilon}.$$

Proof. For $x \in \overline{\{u > 0\}}$ and $\text{dist}(x, D) \geq 1$, $B_1(x) \cap D = \emptyset$ and consequently

$$|\{u > 0\} \cap B_1(x)| \geq \frac{C(n)\epsilon}{\|\phi\|_{C^1} + 1/\epsilon}.$$

By Vitali we reduce $\{B_1(x)\}_{x \in \overline{\{u > 0\}}}$ and $\text{dist}(x, D) \geq 1$ to a disjoint subcollection $\{B_1(x_j)\}_{j \in J}$ with $\{B_5(x_j)\}_{j \in J}$ still covers $\overline{\{u > 0\}} \cap \{\text{dist}(x, D) \geq 1\}$.

Thus one has the following

$$\begin{aligned} m &= |\{u > 0\} \cap D^c| \\ &\geq \sum_J |B_1(x_j) \cap \{u > 0\}| \\ &\geq \frac{C(n)\epsilon}{\|\phi\|_{C^1} + 1/\epsilon} \text{Card}(J). \end{aligned}$$

As a result we have estimate on the cardinality of J . Note that any $x \in \{u > 0\}$ can be connected to $\{\text{dist}(x, D^c) \leq 1\}$ through a chain of at most $\text{Card}(J)$ balls of radius 5, we have the desired estimate. \square

ACKNOWLEDGEMENT

The author would like to thank his PhD advisor, Luis Caffarelli, for many valuable conversations regarding this project. He is also grateful to his colleagues and friends, especially Luis Duque, Dennis Kriventsov and Yijing Wu, for all the discussions and encouragement.

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